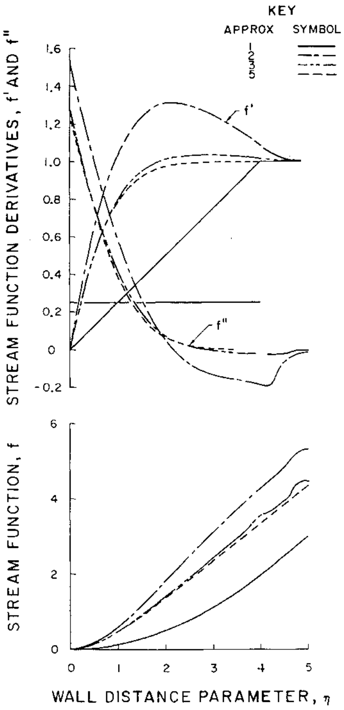

Fig. 1 Quasilinearization solution for the boundary layer on a two-dimensional stagnation point, convergence of approximations for stream function f and its derivatives.

proximations. These types of behavior are probably typical of the convergence process.

No instability is shown by the final profiles, although the velocity comes into its asymptotic value of unity at about 2.5, and the solution was carried out to an η of 5. The "shoot and correct" method of successive approximations mentioned in the Introduction, however, appears to be very sensitive to instability.

The values of all functions in the fifth approximation differed from the corresponding functions in the fourth approximation at each of the 26 tabulated points by less than 5 in the third decimal place. The value of f'' (shear stress) in the fifth approximation was 1.2397 compared with the published value[8] of 1.2326. This accuracy is consistent with the truncation error of 0.001 specified to the integration subroutine. The use of a tighter truncation error bound or convergence criterion (more cycles) and the concomitant additional computer time did not seem justified for the purpose of this test case.

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Orbit Determination by Angular Measurements

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A NEW method is here introduced for the preliminary determination of satellite orbits about the earth. Let $\mathbf{s}$, $\rho\mathbf{p}$ denote the position vectors of the geocenter and an orbiting satellite relative to a tracking station at time t, in which ρ, $\mathbf{p}$ denote slant range and unit vector, respectively. The position of the satellite relative to a geocentered inertial frame is therefore defined by the vector equation $\mathbf{r} = \rho\mathbf{p} - \mathbf{s}$.

Let vector and scalar functions of t_i, $i = 1, 2, 3$, be denoted simply by use of the subscript i, so that $\mathbf{s}(t_i) \triangleq \mathbf{s}_i$, $\rho(t_i) \triangleq \rho_i$. The time intervals will be denoted by Δ_1, Δ_2, Δ_3, so that $\mathbf{r}_2 = \mathbf{r}(t_1 + \Delta_1)$, $\mathbf{r}_3 = \mathbf{r}(t_2 + \Delta_2)$. Let B_{ij} denote the area of the triangle bounded by the vectors $\mathbf{r}_i$, $\mathbf{r}_j$, $\mathbf{r}_j - \mathbf{r}_i$. It follows that $|\mathbf{r}_i \times \mathbf{r}_j| = 2B_{ij}$. Consequently,

$$\frac{\mathbf{r}_1 \times \mathbf{r}_2}{B_{12}} = \frac{\mathbf{r}_2 \times \mathbf{r}_3}{B_{23}} = \frac{\mathbf{r}_1 \times \mathbf{r}_3}{B_{13}} \tag{1}$$

The ratios of the triangular areas are expressible by the formulas of Gibbs (see Ref. 1) which, in the present terminology, are given by

$$\frac{B_{23}}{B_{13}} = \frac{\Delta_2}{\Delta_3}\left[1 + \frac{\mu\Delta_1}{6r_2{}^3}(\Delta_2 + \Delta_3) + \frac{\mu}{12} \times \left(\frac{1}{r_1{}^3} - \frac{1}{r_2{}^3}\right)(\Delta_1\Delta_3 - \Delta_2{}^2) + \ldots\right]$$

$$\frac{B_{12}}{B_{13}} = \frac{\Delta_1}{\Delta_3}\left[1 + \frac{\mu\Delta_2}{6r_2{}^3}(\Delta_1 + \Delta_3) - \frac{\mu}{12} \times \left(\frac{1}{r_1{}^3} - \frac{1}{r_2{}^3}\right)\left(\frac{\Delta_2}{\Delta_1}\right)(\Delta_2\Delta_3 - \Delta_1{}^2) + \ldots\right] \tag{2}$$

The measurements of the azimuth and elevation of a satellite at three instants of time t_1, t_2, t_3 provide sufficient tracking data for a preliminary determination of the orbit of the satellite about the earth. For such a determination, it is convenient to express inertial components of the tracking station position vectors $-\mathbf{s}_i$ ($i = 1, 2, 3$) in terms of the radius of the earth, the station's colatitude and longitude, and the angles of the earth's rotation at times t_i, ($i = 1, 2, 3$). The line-of-sight vectors $\mathbf{p}_i$ ($i = 1, 2, 3$) are computed from the geometry relating two local station-centered reference frames, namely, that in which azimuth and elevation are direction coordinates and that whose axes are parallel to those of the geocentered inertial Cartesian frame. The transformation equations will be omitted here; the essential point of the present discussion is that the components of the vectors $\mathbf{p}_i$, $\mathbf{s}_i$ ($i = 1, 2, 3$) will be assumed known. Since the positions of the satellite are defined by $\mathbf{r}_i = \rho_i\mathbf{p}_i - \mathbf{s}_i$, $i = 1, 2, 3$, only the slant ranges ρ_i have to be calculated to complete the determination of $\mathbf{r}_i$. An iterative process will be described which has been devised for this purpose. For other iterative schemes of the general nature used in this note, see Herget.[2]

The vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are linearly related by the equation

$$B_{23}\mathbf{r}_1 - B_{13}\mathbf{r}_2 + B_{12}\mathbf{r}_3 = 0 \tag{3}$$

In fact, vector multiplications of Eq. (3), first by $\mathbf{r}_2$ and then by $\mathbf{r}_3$, yield Eqs. (1). Conversely, Eqs. (1) imply that the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are coplanar and determine the equation of the

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plane. If, in Eq. (3), the vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are replaced by their forms from $\mathbf{r}_i = \rho_i\mathbf{p}_i - \mathbf{s}_i$ $(i = 1, 2, 3)$, the equation becomes

$$B_{23}\rho_1\mathbf{p}_1 - B_{13}\rho_2\mathbf{p}_2 + B_{12}\rho_3\mathbf{p}_3 = B_{23}\boldsymbol{\sigma} \quad (4)$$

in which

$$B_{23}\boldsymbol{\sigma} = B_{23}\mathbf{s}_1 - B_{13}\mathbf{s}_2 + B_{12}\mathbf{s}_3 \quad (5)$$

The system of equations obtained from the components of vector equation (4) possesses a unique solution ρ_1, ρ_2, ρ_3 if and only if the determinant $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \neq 0$ and not all of the components of the vector $\boldsymbol{\sigma}$ vanish. The solutions are

$$\rho_1 = \frac{(\boldsymbol{\sigma}, \mathbf{p}_2, \mathbf{p}_3)}{D} \qquad \rho_2 = \frac{-B_{23}(\mathbf{p}_1, \boldsymbol{\sigma}, \mathbf{p}_3)}{B_{13}D}$$

$$\rho_3 = \frac{B_{23}(\mathbf{p}_1, \mathbf{p}_2, \boldsymbol{\sigma})}{B_{12}D} \quad (6)$$

in which $D = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$.

In order to apply Eqs. (6) to compute the values of ρ_1, ρ_2, ρ_3, the components of the vector $\boldsymbol{\sigma}$ must be known. Since the components of the vectors $\mathbf{s}_i$ are assumed known, knowledge of the values of the ratios B_{23}/B_{13}, B_{12}/B_{13} will provide sufficient information to complete the determination of σ. The following iterative process serves to determine accurate values of the ratios B_{23}/B_{13}, B_{12}/B_{13}. Since $\mathbf{r}_2 \doteq \mathbf{r}_1 + \dot{\mathbf{r}}_1\Delta_1$, it follows that $1/r_1^3 - 1/r_2^3 \doteq 3\dot{r}_1 \Delta_1/r_1^4$. Consequently, the third terms in the right members of Eq. (2) are of third order in the Δ's. Let $B_{13}^*, B_{12}^*, B_{23}^*$ be so defined that

$$\frac{B_{23}^*}{B_{13}^*} = \frac{\Delta_2}{\Delta_3}\left[1 + \frac{\mu\Delta_1}{6r_2^3}(\Delta_2 + \Delta_3)\right]$$

$$\frac{B_{12}^*}{B_{13}^*} = \frac{\Delta_1}{\Delta_3}\left[1 + \frac{\mu\Delta_2}{6r_2^3}(\Delta_1 + \Delta_3)\right] \quad (7)$$

Let an estimate $\rho_2^{(1)}$ of the slant range ρ_2 be selected and the corresponding value $r_2^{(1)}$ be calculated by means of the equation

$$r_2^{(1)} = |\rho_2^{(1)}\,\mathbf{p}_2 - \mathbf{s}_2| \quad (8)$$

Let $r_2^{(1)}$ be substituted for r_2 in Eqs. (7) and calculate $(B_{23}^*/B_{13}^*)^{(1)}$, $(B_{12}^*/B_{13}^*)^{(1)}$. The first and second equations of (6) with these values substituted for B_{23}/B_{13}, B_{12}/B_{13} yield computed values for ρ_1 and ρ_2, say $\rho_1^{(2)}$ and $\rho_2^{(2)}$. These values are then used to calculate the corresponding values $r_1^{(2)}$, $r_2^{(2)}$ by means of the equations

$$r_1^{(2)} = |\rho_1^{(2)}\mathbf{p}_1 - \mathbf{s}_1| \qquad r_2^{(2)} = |\rho_2^{(2)}\mathbf{p}_2 - \mathbf{s}_2| \quad (9)$$

The values of $r_1^{(2)}$, $r_2^{(2)}$ are then substituted in Eq. (2) to determine ratios denoted by $(B_{12}/B_{13})^{(2)}$ and $(B_{23}/B_{13})^{(2)}$. This iteration process is continued, making use of equations of forms (6, 9, and 2) successively, until the differences $(B_{12}/B_{13})^{(j)} - (B_{12}/B_{13})^{(j+1)}$ and $(B_{23}/B_{13})^{(j)} - (B_{23}/B_{13})^{(j+1)}$ for integer j become negligible. The values of the ratios B_{12}/B_{13}, B_{23}/B_{13} thus determined are used for the final determinations of σ, the slant ranges ρ_1, ρ_2, ρ_3, and position vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}$ in which $\mathbf{r}_i = \rho_i\mathbf{p}_i - \mathbf{s}_i$, $i = 1, 2, 3$.

The orbital elements can be easily calculated in terms of the components of the vectors $\mathbf{r}_i$ and knowledge of tracking station locations (in inertial space) at the instants of time t_1, t_2, t_3. The formulas for the calculations will be presented here without derivations. Let θ_i denote the angle from the line of (ascending) node to the vector $\mathbf{r}_i$ (oriented so that $\theta_1 < \theta_2$). Then

$$\theta_i = \cos^{-1}\left\{\frac{-(\mathbf{r}_i)_1(\mathbf{r}_1 \times \mathbf{r}_2)_2 + (\mathbf{r}_i)_2(\mathbf{r}_1 \times \mathbf{r}_2)_1}{\mathbf{r}_i\,[(\mathbf{r}_1 \times \mathbf{r}_2)_1{}^2 + (\mathbf{r}_1 \times \mathbf{r}_2)_2{}^2]^{1/2}}\right\}(i = 1, 2, 3) \quad (10)$$

in which r_i denotes the magnitude of $\mathbf{r}_i$, and a subscript j $(j = 1, 2)$ of a vector enclosed in parentheses signifies the jth component of the vector. The equation of the orbit may be written in the form

$$\frac{1}{r} = A + B\cos\theta + C\sin\theta \quad (11)$$

in which $A = 1/a\,(1 - e^2)$, $B = e(\cos\theta_p)/a(1 - e^2)$, $C = e(\sin\theta_p)/a(1 - e^2)$ and in which a, e have their usual significations and θ_p is the argument of perigee, usually denoted ω. If the values $\theta_1, \theta_2, \theta_3$ that correspond, by Eq. (10), to three vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are substituted successively into Eq. (11), three equations result which can be readily solved for A, B, C. Let Ω denote the angle from the x axis (of the inertial frame) to the ascending line of nodes. The elements a, e, θ_p are given by the formulas

$$a = \frac{A}{A^2 - (B^2 + C^2)} \qquad e = \frac{(B^2 + C^2)^{1/2}}{A}$$

$$\theta_p = \tan^{-1}\left(\frac{C}{B}\right) \qquad \Omega = \tan^{-1}\left(\frac{(\mathbf{r}_2 \times \mathbf{r}_1)_1}{(\mathbf{r}_1 \times \mathbf{r}_2)_2}\right) \quad (12)$$

The method of orbit determination was programed for double precision computation on a digital computer. The means of testing the accuracies of the method consisted of computing the orbits (all of the orbit parameters) of various hypothetical orbits by using directional data, times of observation, and known positions of tracking stations, and comparing the computed results with the corresponding hypothetical values. A great multiplicity of hypothetical cases were considered in which the elements $\beta, \alpha, a, e, \theta_1, \theta_2, \theta_3, \theta_p, i$ were selected from the following systems: $\beta = 5°, 15°, 25°, 35°, \ldots, 90°$; $\alpha = 4°, 12°, 20°, 28°, \ldots, 68°$; $a = 3922.33, 4443.94, \ldots, 16000$ naut miles; $e = 0, 0.0125, 0.0150, \ldots, 0.85$; $(\theta_1, \theta_2, \theta_3) = (6°, 6.53°, 7.07°), (18°, 18.53°, 19.07°), \ldots, (102°, 102.53°, 103.07°), \ldots, (0°, 2°, 4°), \ldots, (0°, 5°, 6°)$; $\theta_p = -24°, -12°, 0°, 12°, 24°, \ldots, 60°$; $i = 10(j-1)$, $j = 1, 2, 3, \ldots, 13$. The symbols a, e have their usual connotation, β, α denote latitude angle and the angle between the line of ascending node and the meridian plane of the tracking station at the time of the initial observation, and i is the inclination angle of the orbit plane. In addition to the hypothetical systems here enumerated, systems of near critical cases were investigated. The critical case, in which the method is not applicable, occurs when the vectors from the tracking station to the three positions observed at t_1, t_2, t_3 are coplanar. Near-critical cases may occur if the tracking station lies in the orbit plane at the time of one of the three observations. Although many of the near-critical cases required double precision computations, the accuracies achieved were very good. It will suffice, therefore, for the purposes of this report to describe the findings that pertain to the near-critical cases. Multiple parameter families of such cases were considered in which the ranges for the elements were as follows: $\beta = 10°, 20°, \ldots, 90°$; $i = 10°, 20°, \ldots, 120°$; $\alpha = \sin^{-1}(\tan\beta\cot i)$; $\theta_1 = \sin^{-1}(\sin\beta\csc i)$; $\theta_p = -24°, -12°, 0°, \ldots, 60°$; $\theta_2 = \theta_1 + 0.53°$; $\theta_3 = \theta_2 + 0.54°$; $a = 3922.33, \ldots, 16000$ naut miles; $e = 0, 0.0125, 0.0150, \ldots, 0.20$.

The accuracies of the orbit determinations are in general better than those of the following typical near-critical cases. The first value given for each element listed is that of a hypothetical orbit. This is to be compared with the second value that is the computed value based on the given angular data.

Case 1

i = 10°, 10.000069°
a = 3922.32999, 3922.2591 naut miles
e = 0.03, 0.029986499
θ_p = −12°, −12.01017°
ρ_1 = 914.40702, 914.385 naut miles
ρ_2 = 922.4266, 922.4048 naut miles
ρ_3 = 931.58933, 931.5673 naut miles
θ_1 = 18°, 18.001388°
θ_2 = 18.53°, 18.531377°
θ_3 = 19.07°, 19.071367°

Case 2

i = 40°, 39.999986°
a = 3922.32999, 3922.4432

e = 0.01, 0.010020000
θ_p = 60°, 60.118205°
ρ_1 = 779.80816, 779.80865
ρ_2 = 806.28246, 806.28279
ρ_3 = 833.64203, 833.64220
θ_1 = 100°, 99.999972°
θ_2 = 100.53°, 100.529970°
θ_3 = 101.07000°, 101.069968°

Case 3

i = 30°, 30.000009°
a = 16000, 16001.358
e = 0.30, 0.30005244
θ_p = −20°, −20.003833°
ρ_1 = 8113.3207, 8113.3793
ρ_2 = 8126.2608, 8126.3195
ρ_3 = 8139.9188, 8139.977
θ_1 = 10°, 9.99999°
θ_2 = 10.53°, 10.529991°
θ_3 = 11.07°, 11.069992°

Case 4

i = 30°, 29.999965°
a = 16000, 15998.423
e = 0.80, 0.79998041
θ_p = −20°, −20.00039°
ρ_1 = 41.443169, 41.442664
ρ_2 = 44.263139, 44.262486
ρ_3 = 65.061448, 65.060481
θ_1 = 10°, 10.000003°
θ_2 = 10.53°, 10.529995°
θ_3 = 11.07°, 11.069987°

Case 5

i = 30°, 29.999979°
a = −16000, −15998.314
e = 1.50, 1.5000575
ρ_1 = 12143.289, 12143.362
ρ_2 = 12169.763, 12169.836
ρ_3 = 12197.176, 12197.248
θ_1 = 10.0000°, 9.999996°
θ_2 = 10.5300°, 10.599995°
θ_3 = 11.0700°, 11.069995°

Case 6

i = 30°, 29.99990°
a = −16000, −15997.779 naut miles
e = 20.0000, 20.002653
ρ_1 = 351905.91, 351905.91 naut miles
ρ_2 = 353663.25, 353663.25 naut miles
ρ_3 = 355449.83, 355449.83 naut miles
θ_1 = 10.000°, 9.999992°
θ_2 = 10.5300°, 10.529993°
θ_3 = 11.0700°, 11.069992°

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Technical Comments

Heat Transfer at Zero Prandtl Number in Flows with Variable Thermal Properties

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EDWARDS and Tellep[1] have calculated the heat-transfer rate from a fluid with power-law thermal properties in the limit of zero Prandtl number. They solve only the energy equation, since they claim that the nonviscous momentum equation is satisfied by taking the component of velocity parallel to the wall (u) to be everywhere equal to its local external value $u_e(x)$. It is the purpose of this note to point out that this solution of the momentum equation is not correct unless u_e is a constant, so the heat-transfer rates calculated by Edwards and Tellep are good only for the "flat-plate" case. Furthermore, for this case, a simple approximate analytical expression can be derived for the heat-transfer rate, which can be fitted to the exact calculations (for a cold wall) within ±3%.

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In the von Mises variables x, ψ the nonviscous momentum equation is

$$u\frac{\partial u}{\partial x} - \frac{\rho_e}{\rho}u_e\frac{du_e}{dx} = 0 \tag{1}$$

It is clear that $u = u_e(x)$ is a solution only if $\rho = \rho_e(x)$ or if u_e is constant. The former case is the incompressible fluid previously treated by many authors, but does not correspond to the variable-property fluid treated by Edwards and Tellep. For variable properties, only u_e = const satisfies the momentum equation independent of the solution of the energy equation, and so it is only to this case that the work of Edwards and Tellep applies. For variable u_e and variable properties, the momentum and energy equations are coupled, and solution is made difficult, even for similarity cases, by a singularity at the wall. The physical reason for the coupling is easy to see. If the temperature varies in the boundary layer, so does the density, and that effects the momentum balance through the inertia terms, even if the viscous terms are neglected.

Edwards and Tellep's results are thus applicable to the flat-plate geometry in the zero Prandtl number limit. This is identical with the case recently studied by Jepson,[2] as well as the end wall case in a shock tube recently studied by Fay and Kemp.[3] Following an idea of Jepson,[2] the author has derived by analytical means an approximate expression for the heat-transfer rate to a cold wall which can be fitted very accurately to exact solutions.[4] This expression is, in the notation of Ref. 1,

$$q = [(u_e/2x)\rho_e k_e c_{pe}]^{1/2}(T_e - T_0)Q \tag{2}$$

$$Q \equiv \frac{1}{(2)^{1/2}}\frac{\rho_w k_w}{\rho_e k_e}\left(\frac{d}{d\eta}\frac{T - T_0}{T_e - T_0}\right)_w = \frac{\Omega'}{(2)^{1/2}(r + s + 1)} \tag{3}$$